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FLEXURAL VIBRATION AND BUCKLING ANALYSIS OF ORTHOTROPIC PLATES BY THE BOUNDARY ELEMENT METHOD

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Abstract—A numerical solution technique by the boundary element method is developed in this paper for the flexural vibration and buckling analysis of elastic orthotropic plates according to Kirchhoff's theory. The integral formulations of the problem make use of the same fundamental solution as for the bending of orthotropic plates. An assumed unknown transverse distributed loading, defined inside the plate domain, is introduced for representing the inertia forces of vibration and the in-plane forces of buckling of the plate. The integral equations necessary for solving the problem of interest, are derived from the integral representations developed earlier for the bending analysis of orthotropic plates. A simple discretization scheme for the plate boundary and its interior domain is adopted in this paper for establishing the integral equations thus obtained in matrix form. After elimination of the conventional boundary unknowns, the flexural vibration or the buckling problem of an orthotropic plate is finally reduced into an eigenvalue problem of a square matrix. The eigenvalues and eigenvectors of that square matrix correspond respectively to the frequencies and the deflection mode shapes of the flexural vibration problem, or to the critical loads and the curvature mode shapes of the buckling problem. Several computational examples of vibration and buckling problems with various boundary conditions are presented, and the numerical results demonstrate, in comparison with some published results, a satisfactory accuracy of the proposed method.

1. INTRODUCTION

The problems of orthotropic plates encountered in engineering have attracted the attention of many researchers, since such structural components are becoming more and more popular due to the increased use of composite materials in modern technology (Tsai and Hahn, 1980; Laroze and Barrau, 1987). Among the numerical techniques commonly employed for solving practical plate problems such as bending, vibration, buckling etc., the finite element method is now well recognized as an extremely versatile and powerful tool. Other numerical methods have more recently been developed as alternative approaches to treat the plate boundary-value problems. In particular, the Boundary Integral Equation Method (BIEM), or Boundary Element Method (BEM) is increasingly becoming an accepted numerical technique which could provide some competitive advantages over the finite element method for the solution of some engineering problems (Banerjee and Butterfield, 1981).

Application of boundary element methods to plate problems can be by two different approaches, that is, the indirect method (IBEM) and the direct method (DBEM). In the earliest work on the subject, the IBEM had been widely explored. A summary of the integral formulations of this method can be found in the book by Jaswon and Symm (1977). It was however on the DBEM that the most important developments have been made during the last decade for the numerical solution of plate problems, particularly for the bending of elastic isotropic plates (for example Bézine, 1978; Stern, 1979; Du *et al.*, 1984). One of the most interesting features of the DBEM approach is that the integral formulations involve conventional boundary variables such as deflection, normal slope, bending and twisting moment and equivalent shear force, upon which the usual boundary conditions are imposed in Kirchhoff bending theory. Some extensions of the direct boundary element method to bending of isotropic plates on elastic foundation, and to free vibration and buckling problems, have more recently been accomplished by Bézine (1980, 1988), Costa and Brebbia

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(1985), and Bézine *et al.* (1985). Although the DBEM formulations are now well established for thin isotropic plates, their application to the analysis of orthotropic plates has, however, only been attempted a few times up until now. Some preliminary work seems to be that of Wu and Altiero (1981) who proposed a solution procedure, based on the influence function method, for the bending of anisotropic plates. Other results have been reported by Kamiya and Sawaki (1982) for the orthotropic plate bending problem by using a simplified DBEM formulation. More recently, Shi and Bézine (1987, 1988) developed a general numerical solution technique by applying the DBEM for the bending analysis of Kirchhoff anisotropic plates. As the integral formulations are derived directly from the generalized Rayleigh–Green identity after having taken into account the possible corners on the plate boundary, this new DBEM solution technique can be efficiently applied to treat practical bending problems of anisotropic plates with arbitrary planforms, whatever the imposed boundary conditions of the problem.

On the other hand, the flexural vibration and buckling problems of orthotropic plates have not, to the author's knowledge, yet been solved by use of the boundary element method, at least by the DBEM technique. This is probably owing to the difficulties in obtaining the fundamental solutions associated with the differential equations of such problems. In fact, the implementation of BEM always necessitates an appropriate fundamental solution of the problem under consideration. For the vibration or buckling problem of orthotropic plates, this fundamental solution is not known so far, and it would be quite difficult, even impossible, to find out it for the general cases. Nevertheless, the flexural vibration and buckling problems of orthotropic plates have been investigated by using other methods rather than the DBEM, and one can find some interesting results in the literature. For example, the series-type method has been employed by different authors (Dickinson, 1969; Narita, 1981; Narita *et al.*, 1982) for studying the free vibration of orthotropic plates. Other techniques such as the edge-function method (O'Callaghan and Studdert, 1985), and the Lagrangian multiplier (Ramkumar *et al.*, 1987), have also been developed for the solution of vibration problems. In the case of buckling analysis, Simitses and Giri (1977) proposed a solution procedure using the modified Galerkine method to predict the critical conditions for rotationally restrained orthotropic plates loaded by a uniform axial compression. More recently, the buckling problem of an orthotropic plate under biaxial loadings has been discussed by Tung and Surdenas (1987) for the case of simply supported boundary conditions.

The published papers mentioned above, dealing with the vibration or the buckling problem, are often limited to some special cases. This is because the numerical procedures employed in those papers are almost based on the so-called series-type method which uses the developments on series of special functions (trigonometrical functions, for instance). In order that the prescribed boundary conditions might be more easily satisfied by the supposed series-functions, the plate planforms and the edge conditions should be restricted to some particular cases such as the simply-supported or clamped square plates.

In this paper, a general DBEM technique is presented for the numerical solution of free flexural vibration and buckling problems of thin orthotropic plates. Based upon the previous work for the bending analysis of orthotropic plates (Shi and Bézine, 1987, 1988), the integral formulations of the vibration and buckling problems are established by utilizing the same fundamental solution as for the bending analysis. The starting point of the solution procedure consists in considering the inertia forces (vibration) and the in-plane forces (buckling) of the plate as an unknown transverse loading distributed inside the plate domain. By introduction of this unknown distributed loading into the integral representation for the bending of orthotropic plates, one obtains three integral equations which involve the conventional boundary variables (deflection, normal slope, bending and twisting moments, shear force), and as well the unknown distributed loading. The numerical formulations of the problem are carried out in this paper by a simple discretization scheme of constant elements for both the plate boundary and its interior domain. By elimination of the conventional boundary unknowns in the matrix formulation, one finally transforms the vibration or the buckling problem into an equivalent eigenvalue problem of a square matrix. The calculation of the frequencies and mode shapes of vibration, or of the critical loads and corresponding mode shapes of buckling can then be easily performed by determining the eigenvalues and the eigenvectors of this derived square matrix. To illustrate the nature of the proposed numerical method, several application examples of flexural vibration and buckling problems are presented, including some elastic orthotropic plates with different boundary conditions (clamped and simply-supported plates, cantilever plate, mixed conditions).

2. BASIC EQUATIONS

Consider a thin elastic orthotropic plate of uniform thickness h and with an arbitrary planform in the Cartesian coordinate system O : xyz. Let Ω be the bounded domain occupied by the plate in the coordinate Oxy-plan; its boundary is denoted by Γ on which N possible corner points A_i of abscissa s_i ($i = 1, \ldots, N$) could be embodied. The two symmetry axes of the orthotropic material are assumed to be parallel to the x- and y-axis, respectively. The flexural rigidities of such a plate can be given by

$$\left.\begin{aligned} &D_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}, \quad D_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}, \quad D_{66} = \frac{Gh^3}{12} \\ &D_{12} = \nu_1 D_{22} \quad \text{or} \quad D_{12} = \nu_2 D_{11}, \quad D_3 = D_{12} + 2D_{66} \end{aligned}\right\} \tag{1}$$

where E_1 and E_2 are the Young's moduli, ν_1 and ν_2 are the Poisson's ratios along, respectively, the two principal directions of the material orthotropy, and G is the shear modulus.

We will describe in the following the basic formulations of the flexural vibration and buckling analysis of orthotropic plates according to Kirchhoff's theory (Lekhnitskii, 1968). Firstly, for the flexural vibration of an orthotropic plate, the problem is governed by the differential equation:

$$D_{11}\frac{\partial^4 W}{\partial x^4} + 2D_3\frac{\partial^4 W}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 W}{\partial y^4} + \rho h\frac{\partial^2 W}{\partial t^2} = 0 \quad \forall\,(x,y)\in\Omega \quad \text{and}\, t\in(0,\infty) \tag{2}$$

where ρ is the mass density per unit volume of the orthotropic plate material, and $W = W(x, y, t)$ represents the transverse deflection at an arbitrary point of coordinates (x, y) on the middle surface of the plate at any moment t during the vibration motion. When the plate under consideration is only undergoing the free vibrations, the deflection $W(x, y, t)$ should be a harmonic function of the time t. Consequently, the deflection solution of governing eqn (2) can be expressed in the following form

$$W(x,y,t) = (A\cos\omega t + B\sin\omega t)w(x,y) \tag{3}$$

where ω is the circular frequency of the plate; $w(x, y)$ is the corresponding mode shape (deflection) at the given instant t; and the constants A and B could be determined by the initial conditions (the given initial deflection and the velocity of the plate, for example) of the problem. By substituting (3) into eqn (2), one obtains the following equation for the deflection $w(x, y)$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = \rho h\omega^2\cdot w(x,y) \quad \forall\,(x,y)\in\Omega. \tag{4}$$

The free vibration problem should be solved by coupling the differential eqn (4) with some suitable homogeneous boundary conditions to be given below. The solution procedure consists of determining the deflection $w(x, y)$ (mode shape) and the vibration frequency ω.

In the case of orthotropic plate buckling, the problem can be formulated in a similar manner. Suppose that the plate under consideration is subjected to the in-plane forces N_x, N_y and N_{xy}, directed to the x-, y-axis, and clockwise respectively. According to Kirchhoff's theory (Lekhnitskii, 1968), the deflection function $w(x, y)$, resulting from the in-plane forces above, should satisfy the following differential equation

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+N_y\frac{\partial^2 w}{\partial y^2}\quad\forall\,(x,y)\in\Omega. \tag{5}$$

For the classical buckling analysis, we have to determine a critical buckling load N_{cr}, for which the orthotropic plate is in a critical state of equilibrium with the corresponding mode shape $w(x, y)$. Generally, we suppose that the given in-plane forces N_x, N_y and N_{xy} maintain respectively a ratio a_x, a_y and a_{xy} with respect to the critical load N_{cr}, i.e.

$$N_x = N_{cr}a_x,\quad N_y = N_{cr}a_y\quad\text{and}\quad N_{xy} = N_{cr}a_{xy}. \tag{6}$$

Then, the differential eqn (5) can be rewritten as follows

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=N_{cr}\left(a_x\frac{\partial^2 w}{\partial x^2}+2a_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+a_y\frac{\partial^2 w}{\partial y^2}\right)\quad\forall\,(x,y)\in\Omega. \tag{7}$$

For the sake of simplicity, we introduce the differential operator $\Psi(.)$ defined in the interior domain Ω by

$$\Psi(.) = D_{11}\frac{\partial^4(.)}{\partial x^4}+2D_3\frac{\partial^4(.)}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4(.)}{\partial y^4}\quad\forall\,(x,y)\in\Omega. \tag{8}$$

Furthermore, we introduce an equivalent force $q(x, y)$ for representing the inertia force of vibration ($\rho h\omega^2 w$) in eqn (4), or the in-plane forces of buckling in eqn (7):

$$q(x,y) = \lambda\cdot\mathbb{P}(w) \tag{9}$$

where $\mathbb{P}(.)$ is an operator depending on the deflection function $w(x, y)$, and λ is an unknown constant, defined by

$$\mathbb{P}(w) = w,\quad \lambda = \rho h\omega^2 \tag{10}$$

for the vibration problem; and

$$\mathbb{P}(w) = a_x\frac{\partial^2 w}{\partial x^2}+2a_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+a_y\frac{\partial^2 w}{\partial y^2},\quad \lambda = N_{cr} \tag{11}$$

for the buckling problem. Note that in the buckling problem, the operator $\mathbb{P}(.)$ represents in fact the curvatures of the deformed orthotropic plate.

By employing the operator (8) and the definitions (10) and (11), one can recast the differential equations of vibration (4) and of buckling (7) in the same simplified form as follows

$$\Psi[w(x,y)] = \lambda\cdot\mathbb{P}[w(x,y)]\quad\forall\,(x,y)\in\Omega. \tag{12}$$

Whether the problem is one of vibration or of buckling, the deflection $w(x, y)$ should satisfy some homogeneous boundary conditions at the plate edges. Within the framework of Kirchhoff's theory, these boundary conditions can be commonly grouped in the following forms:

—clamped edge:

$$w(x,y) = 0,\quad \Theta_n[w(x,y)] = 0\quad\forall\,(x,y)\in\Gamma \tag{13}$$

—simply-supported edge:

$$w(x,y) = 0, \quad M_n[w(x,y)] = 0 \quad \forall\,(x,y)\in\Gamma \tag{14}$$

—free edge:

$$M_n[w(x,y)] = 0, \quad V_n[w(x,y)] = 0 \quad \forall\,(x,y)\in\Gamma \tag{15}$$

where $\Theta_n(.)$, $M_n(.)$ and $V_n(.)$ represent the operators defined on the boundary Γ which correspond respectively to the normal slope, the bending moment and the equivalent shear force. With the addition of twisting moment operator $T_n(.)$, those boundary operators can be generally defined as (Shi, 1989):

$$\Theta_n(.) = \frac{\partial(.)}{\partial n} = \cos\alpha\frac{\partial(.)}{\partial x} + \sin\alpha\frac{\partial(.)}{\partial y} \tag{16}$$

$$M_n(.) = -\left[(D_{11}\cos^2\alpha + D_{12}\sin^2\alpha)\frac{\partial^2(.)}{\partial x^2} + 4D_{66}\sin\alpha\cos\alpha\frac{\partial^2(.)}{\partial x\,\partial y} + (D_{12}\cos^2\alpha + D_{22}\sin^2\alpha)\frac{\partial^2(.)}{\partial y^2}\right] \tag{17}$$

$$\begin{aligned} V_n(.) = -\Big\{&\cos\alpha[D_{11}(1+\sin^2\alpha) - D_{12}\sin^2\alpha]\frac{\partial^3(.)}{\partial x^3} \\ &+\sin\alpha[D_{12}(1+\cos^2\alpha) + 4D_{66}\sin^2\alpha - D_{11}\cos^2\alpha]\frac{\partial^3(.)}{\partial x^2\,\partial y} \\ &+\cos\alpha[D_{12}(1+\sin^2\alpha) + 4D_{66}\cos^2\alpha - D_{22}\sin^2\alpha]\frac{\partial^3(.)}{\partial x\,\partial y^2} \\ &+\sin\alpha[D_{22}(1+\cos^2\alpha) - D_{12}\cos^2\alpha]\frac{\partial^3(.)}{\partial y^3}\Big\} \end{aligned} \tag{18}$$

$$T_n(.) = -\left[(D_{12} - D_{11})\sin\alpha\cos\alpha\frac{\partial^2(.)}{\partial x^2} + 2D_{66}(\cos^2\alpha - \sin^2\alpha)\frac{\partial^2(.)}{\partial x\,\partial y} + (D_{22} - D_{12})\cos\alpha\sin\alpha\frac{\partial^2(.)}{\partial y^2}\right] \tag{19}$$

where $\mathbf{n}$ represents the outward normal at a regular point of the boundary Γ, and α is the angle from the x-axis to the normal $\mathbf{n}$.

The problem posed by the governing eqn (12) and a couple of boundary conditions to be chosen from (13)–(15) following the problem type (which is in fact the vibration or the buckling problem), can be solved by using the direct boundary element method. The application of this DBEM to orthotropic plate bending has been investigated quite fully by Shi and Bézine (1987, 1988). Hence, in the present study of the vibration and buckling problems, it is possible to establish the necessary integral equations by applying the representations devised for the bending problem of orthotropic plates.

3. INTEGRAL FORMULATION

The integral formulations of plate bending problems by the DBEM solution technique were firstly derived from the Rayleigh–Green identity, which corresponds at its origin to the so-called bilinear form for the isotropic plates (Bergman and Schiffer, 1953). The recent developments of the DBEM, during the last 10 years, required an adequate treatment for some plates having corner points at their boundary. It was just under such an impetus that several researchers had established the generalized Rayleigh–Green identity for the bending of isotropic plates with arbitrary planforms (for example Bézine, 1978; Stern, 1979). In the case of anisotropic plates, this generalized Rayleigh–Green identity has been obtained by Shi (1989) by applying the reciprocal theorem of linear elasticity, and application to bending problems has been reported by Shi and Bézine (1987, 1988).

3.1. *Integral equations*

The generalized Rayleigh–Green identity for orthotropic plates can be expressed as follows

$$\int_\Gamma [V_n(u)w - M_n(u)\Theta_n(w) + \Theta_n(u)M_n(w) - uV_n(w)]\,ds + \sum_{i=1}^{N} [\![T_n(u)w - uT_n(w)]\!]_{A_i} = \int_\Omega [u\Psi(w) - \Psi(u)w]\,dS \quad (20)$$

where $u(x,y)$ and $w(x,y)$ are two arbitrary functions which should be four times continuously differentiable inside the domain Ω and three at the boundary Γ; $\Psi(.)$ is the operator defined by (8); and $[\![.]\!]$ represents the discontinuity jump at the N corner points A_i of abscissa s_i $(i = 1, \ldots, N)$ at the boundary Γ:

$$[\![\cdot]\!]_{A_i} = (\cdot)|_{s_i^+} - (\cdot)|_{s_i^-}. \quad (21)$$

The integral formulations of the bending problem have been obtained from the generalized Rayleigh–Green identity (20) by introducing the corresponding fundamental solution (Shi and Bézine, 1987, 1988). In the present study, if we consider the inertia force of vibration, or the in-plane forces of buckling, as an unknown transverse loading, the problem of interest could be then treated, according to eqn (12), as a bending one. In other words, the integral equations of the vibration or of the buckling problem can be established by substitution of the unknown distributed loading $q = \lambda \mathbb{P}(w)$ into the integral representations previously developed for the bending analysis. Therefore, the fundamental solution, say $w^s(Q;P)$, for the vibration or the buckling problem can be the same as for the bending of orthotropic plates, which satisfies

$$\Psi[w^s(Q;P)] = \delta(Q;P) \quad (22)$$

where Q and P represent respectively the distribution point and the source point; and $\delta(Q;P)$ is the Dirac δ-function with origin at the source point P. The complete expression of the fundamental solution $w^s(Q;P)$ will be given in the next subsection.

In the following, we summarize the principal integral equations for the vibration and buckling problems, governed by eqn (12) and two corresponding boundary conditions among (13)–(15). These integral equations have been obtained simply by replacing the equivalent loading $q = \lambda \cdot \mathbb{P}(w)$ of (9) into the bending integral representations (Shi and Bézine, 1988).

—Boundary integral equations:

$$\tfrac{1}{2}w(P) + \int_\Gamma [V_n(w^s)w - M_n(w^s)\Theta_n(w) + \Theta_n(w^s)M_n(w) - w^sV_n(w)]\,ds + \sum_{i=1}^{N} [\![T_n(w^s)w - w^sT_n(w)]\!]_{A_i} = \int_\Omega w^s \cdot \lambda \mathbb{P}(w)\,dS \quad \forall P \in \Gamma \quad (23)$$

$$\tfrac{1}{2}\Theta_n[w(P)]+\int_\Gamma\left[\frac{\partial V_n(w^s)}{\partial n_0}w-\frac{\partial M_n(w^s)}{\partial n_0}\Theta_n(w)+\frac{\partial \Theta_n(w^s)}{\partial n_0}M_n(w)-\frac{\partial w^s}{\partial n_0}V_n(w)\right]\mathrm{d}s$$

$$+\sum_{i=1}^{N}\left[\!\left[\frac{\partial T_n(w^s)}{\partial n_0}w-\frac{\partial w^s}{\partial n_0}T_n(w)\right]\!\right]_{A_i}=\int_\Omega\frac{\partial w^s}{\partial n_0}\cdot\lambda\mathbb{P}(w)\,\mathrm{d}S\quad\forall P\in\Gamma \quad (24)$$

where $\mathbf{n}_0$ is the outward normal at source point P on the boundary Γ.

—Integral representation of deflection inside the domain:

$$w(P)=\int_\Omega w^s\cdot\lambda\mathbb{P}(w)\,\mathrm{d}S-\int_\Gamma[V_n(w^s)w-M_n(w^s)\Theta_n(w)+\Theta_n(w^s)M_n(w)-w^sV_n(w)]\,\mathrm{d}s$$

$$-\sum_{i=1}^{N}[\![T_n(w^s)w-w^sT_n(w)]\!]_{A_i}\quad\forall P\in\Omega. \quad (25)$$

The boundary integral eqns (23), (24) and the integral representation of deflection (25) involve four fundamental variables defined on Γ, that is, the deflection w, the normal slope $\Theta_n(w)$, the bending moment $M_n(w)$ and the equivalent shear force $V_n(w)$. Two of those boundary variables should be given by the boundary conditions to be chosen from (13)–(15); and the two others are the conventional unknowns of the problem. Moreover, it should be noted that the twisting moment $T_n(w)$, appearing also in the integral equations (23)–(25) by means of the sum of jumps at the corner points A_i ($i = 1,\ldots,N$) on the boundary Γ, is not taken as a fundamental variable in the present method. In fact, it will be shown that the twisting moment $T_n(w)$ could be expressed in terms of w, $\Theta_n(w)$ and $M_n(w)$ in the neighborhood of the corner points by the technique that we will discuss later in the section.

For the problem of free vibration, the inertia force, denoted by $q = \lambda\mathbb{P}(w)$, is proportional to the deflection $w(x,y)$ since $\lambda\mathbb{P}(w) = \rho h\omega^2 w$ according to (10). So we have in total three unknowns in eqns (23), (24) and (25) for such a problem, i.e. the two conventional boundary unknowns, and the unknown distributed loading (inertia force) $\lambda\mathbb{P}(w) = \rho h\omega^2 w$ inside the domain Ω. However, in the case of buckling problem, the equivalent loading $q = \lambda\mathbb{P}(w)$, with $\mathbb{P}(w)$ defined in (11), depends on the curvature of the deformed plate. Consequently, the integral representation (25) of deflection $w(P)$ could not be used directly to solve the buckling problem together with the boundary integral eqns (23) and (24). So it is necessary to establish a supplementary integral representation which could link up the curvature $\mathbb{P}(w)$ with the boundary variables w, $\Theta_n(w)$, $M_n(w)$ and $V_n(w)$.

—Integral representation of operator $\mathbb{P}(.)$ inside the domain:

Note that the integral representation (25) of deflection $w(P)$ with $P\in\Omega$ is sufficiently regular and differentiable twice inside the plate domain Ω. Therefore, it is possible to extend the operator $\mathbb{P}(.)$ of the plate curvature, defined by (11) for the buckling problem, to the deflection $w(P)$ in (25). The integral representation thus obtained for the operator $\mathbb{P}(.)$ can be expressed as follows

$$\mathbb{P}[w(P)]=\int_\Omega\mathbb{P}(w^s)\cdot\lambda\mathbb{P}(w)\,\mathrm{d}S-\int_\Gamma\{\mathbb{P}[V_n(w^s)]w-\mathbb{P}[M_n(w^s)]\Theta_n(w)+\mathbb{P}[\Theta_n(w^s)]M_n(w)$$

$$-\mathbb{P}(w^s)V_n(w)\}\,\mathrm{d}s-\sum_{i=1}^{N}[\![\mathbb{P}[T_n(w^s)]w-\mathbb{P}(w^s)T_n(w)]\!]_{A_i}\quad\forall P\in\Omega. \quad (26)$$

This integral representation can be effectively considered to be valid for both the buckling and the vibration problem, because in the case of vibration, representation (26) becomes an identical one to that of the deflection (25) according to the definition (10) of the operator $\mathbb{P}(.)$ in this case.

The boundary integral eqns (23) and (24) together with the integral representation (26) are sufficient to solve simultaneously the three unknowns, that is, the two conventional boundary unknowns and the unknown distributed loading $\lambda \mathbb{P}(w)$, of the problem, whether it is a vibration or a buckling one.

3.2. *Fundamental solution*

The fundamental solution, also named as singular solution or Green's function in the literature, plays an important role in the applications of BEM technique. In the case of orthotropic plates, this fundamental solution was found for the first time by Mossakowski (1954). One year later, the same author also gave the fundamental solution for the bending of anisotropic plates (Mossakowski, 1955). The solution procedure to obtain this fundamental solution for the general case of anisotropic plates has been investigated by Suchar (1964), by using the complex functions, for both the concentrated force and the concentrated moment.

The fundamental solution of orthotropic plates in flexure represents the deflection solution of an infinite plate. This infinite plate, made up of the same material as for the orthotropic plate under consideration, is subjected to a concentrated unit force applied at the source point $P(x_0, y_0)$. Let $w^s(Q;\ P)$ be the deflection of the infinite plate at the distribution point $Q(x,y)$. Then the function $w^s(Q;\ P)$, say the fundamental solution, should satisfy eqn (22):

$$D_{11}\frac{\partial^4 w^s(Q;P)}{\partial x^4} + 2D_3\frac{\partial^4 w^s(Q;P)}{\partial x^2\,\partial y^2} + D_{22}\frac{\partial^4 w^s(Q;P)}{\partial y^4} = \delta(Q;P). \tag{27}$$

The corresponding characteristic equation of eqn (27) can therefore be written as follows

$$D_{22}\mu^4 + 2D_3\mu^2 + D_{11} = 0. \tag{28}$$

As shown by Lekhnitskii (1968), the algebraical eqn (28) cannot have real roots for a homogeneous elastic plate. So one can express the four complex roots of eqn (28) in the following form

$$\mu_{1,2} = d_1 \pm ie_1 \quad \text{and} \quad \mu_{3,4} = d_2 \pm ie_2 \quad (e_1 > 0, e_2 > 0). \tag{29}$$

Three cases of the roots of eqn (28) should be discussed.

Case 1, $D_3^2 > D_{11}D_{22}$:

$$\begin{aligned} &d_1 = d_2 = 0\\ &e_1 = \sqrt{\frac{D_3 - \sqrt{D_3^2 - D_{11}D_{22}}}{D_{22}}}, \quad e_2 = \sqrt{\frac{D_3 + \sqrt{D_3^2 - D_{11}D_{22}}}{D_{22}}}. \end{aligned} \tag{30}$$

Case 2, $D_3^2 = D_{11}D_{22}$:

$$\left.\begin{aligned} d_1 &= d_2 = 0\\ e_1 &= e_2 = \sqrt{\frac{D_3}{D_{22}}} \end{aligned}\right\} \tag{31}$$

Case 3, $D_3^2 < D_{11}D_{22}$:

$$\left.\begin{aligned} d_1 &= -d_2 = \sqrt{\frac{\sqrt{D_{11}D_{22}} - D_3}{2D_{22}}}\\ e_1 &= e_2 = \sqrt{\frac{\sqrt{D_{11}D_{22}} + D_3}{2D_{22}}} \end{aligned}\right\} \tag{32}$$

In Case 1 and Case 3 ($D_3^2 \neq D_{11}D_{22}$), the fundamental solution $w^s(Q;P)$ of eqn (27) can be written, according to the survey of Mossakowski (1954) and Suchar (1964), in the following form

$$w^s(Q;P) = \frac{1}{8\pi D_{22}} \sum_{i=1}^{2} [A_i R_i(Q;P) - B_i S_i(Q;P)] \tag{33}$$

where $R_i(Q;P)$ and $S_i(Q;P)$, with $i = 1$ and 2, are two functions given by

$$R_i(Q;P) = (x_i^2 - y_i^2)\left[\ln\frac{x_i^2+y_i^2}{a^2} - 3\right] - 4x_i y_i \arctan\frac{y_i}{x_i} \tag{34}$$

$$S_i(Q;P) = x_i y_i\left[\ln\frac{x_i^2+y_i^2}{a^2} - 3\right] + (x_i^2+y_i^2)\arctan\frac{y_i}{x_i} \tag{35}$$

with x_i and y_i ($i = 1$ and 2) defined by the distribution point $Q(x, y)$ and the source point $P(x_0, y_0)$:

$$x_i = (x-x_0) + d_i(y-y_0), \quad y_i = e_i(y-y_0) \quad (i = 1, 2). \tag{36}$$

The two constants A_i and B_i ($i = 1, 2$) in the fundamental solution (33) are determined by

$$A_i = \frac{(d_1-d_2)^2 + (-1)^i(e_1^2 - e_2^2)}{Ce_i} \tag{37}$$

$$B_i = \frac{4(-1)^i(d_1-d_2)}{C} \tag{38}$$

with

$$C = (d_1-d_2)^4 + 2(d_1-d_2)^2(e_1^2+e_2^2) + (e_1^2-e_2^2)^2. \tag{39}$$

The fundamental solution $w^s(Q;P)$ involves a normalization coefficient, a, which appears in the functions $R_i(Q;P)$ and $S_i(Q;P)$. It has been proved in practice that such a coefficient does not affect the numerical results of the present DBEM technique. So one can always choose that $a = 1$, for example.

In the case of $D_3^2 = D_{11}D_{22}$, the fundamental solution $w^s(Q;P)$ given in (33) becomes an indeterminate expression. For such a case, Mossakowski (1954) has demonstrated that the fundamental solution can be expressed as follows:

$$w^s(Q;P) = \frac{1}{16\pi\varepsilon^3 D_{22}}\left\{[(x-x_0)^2+\varepsilon^2(y-y_0)^2]\ln\left[\frac{(x-x_0)^2+\varepsilon^2(y-y_0)^2}{a^2}\right] - [3(x-x_0)^2+\varepsilon^2(y-y_0)^2]\right\} \tag{40}$$

where the real coefficient ε is determined by

$$\varepsilon^4 = \frac{D_{11}}{D_{22}}. \tag{41}$$

The kernel functions appearing in the integral eqns (23)–(26) can be obtained by the successive derivations of the fundamental solution $w^s(Q;P)$ given in (33) or (40). For the case of anisotropic plates, the authors have given all the derivatives of this fundamental solution from the first to the fourth order (Shi and Bézine, 1988). The fifth-order derivatives needed in integral representation (26) for the buckling analysis will be given in the Appendix.

3.3. *Treatment of twisting moment*

As remarked above, the twisting moment $T_n(w)$ is not considered, in the present DBEM solution technique, as a fundamental variable. To evaluate the sum of jumps in the integral eqns (23)–(26), one can always express the twisting moment $T_n(w)$ in terms of the fundamental boundary variables w, $\Theta_n(w)$ and $M_n(w)$ in the neighborhood of corners A_i ($i = 1, \ldots, N$) on the boundary Γ.

In fact, the boundary operators $M_n(.)$ and $T_n(.)$, given in (17) and (19), associated with the deflection $w(x, y)$ can be rewritten in the following form

$$M_n(w) = -\left(f_1 \frac{\partial^2 w}{\partial n^2} + f_2 \frac{\partial^2 w}{\partial n\, \partial t} + f_3 \frac{\partial^2 w}{\partial t^2}\right) \tag{42}$$

$$T_n(w) = -\left(g_1 \frac{\partial^2 w}{\partial n^2} + g_2 \frac{\partial^2 w}{\partial n\, \partial t} + g_3 \frac{\partial^2 w}{\partial t^2}\right) \tag{43}$$

where **t** represents the tangent of the boundary Γ; and the coefficients f_1, f_2, f_3 and g_1, g_2, g_3 are given by

$$\begin{aligned} f_1 &= D_{11} \cos^4 \alpha + 2D_3 \sin^2 \alpha \cos^2 \alpha + D_{22} \sin^4 \alpha \\ f_2 &= [D_{22} \sin^2 \alpha - D_{11} \cos^2 \alpha + D_3 (\cos^2 \alpha - \sin^2 \alpha)] \sin 2\alpha \\ f_3 &= D_{12} + (D_{11} + D_{22} - 2D_3) \sin^2 \alpha \cos^2 \alpha \end{aligned} \tag{44}$$

and

$$\begin{aligned} g_1 &= \tfrac{1}{2} f_2 \\ g_2 &= 2D_{66} + 2(D_{11} + D_{22} - 2D_3) \sin^2 \alpha \cos^2 \alpha \\ g_3 &= [D_{22} \cos^2 \alpha - D_{11} \sin^2 \alpha - D_3 (\cos^2 \alpha - \sin^2 \alpha)] \sin 2\alpha. \end{aligned} \tag{45}$$

By eliminating the term $\partial^2 w / \partial n^2$ in (42) and (43), and by employing the normal slope operator $\Theta_n(.)$ given in (16), one obtains the following expression

$$T_n(w) = \frac{1}{f_1}\left[g_1 M_n(w) + (g_1 f_2 - g_2 f_1) \frac{\partial \Theta_n(w)}{\partial t} + (g_1 f_3 - g_3 f_1) \frac{\partial^2 w}{\partial t^2} \right]. \tag{46}$$

After discretization of the plate boundary Γ (the discretization procedure will be described in detail in the next section), it is then possible to approximate the twisting moment $T_n(w)$ at a corner point A_i ($i = 1, \ldots, N$) by the values of $M_n(w)$, $\Theta_n(w)$ and w at some nodes near this corner:

$$T_n(w)|_{A_i} \Leftarrow \langle M_n(w), \Theta_n(w), w \rangle|_{\text{nodal values}}. \tag{47}$$

For example, we consider three boundary elements K_j, K_{j+1} and K_{j+2} of equal length l near

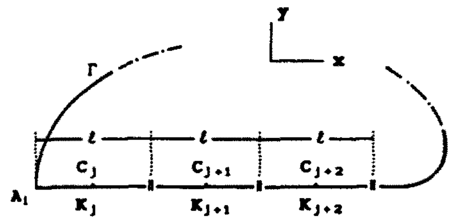

Fig. 1. Treatment near a boundary corner.

the corner point A_i $(i = 1, \ldots, N)$ of the boundary Γ (see Fig. 1). The corresponding nodes (middle point of each segment) of those three elements are designated by C_j, C_{j+1} and C_{j+2}, respectively. Consequently, one can interpolate the terms of (46) in the following way

$$M_n(w)|_{s_i^+} \cong M_n[w(C_j)] \tag{48}$$

$$\left.\frac{\partial \Theta_n(w)}{\partial t}\right|_{s_i^+} \cong \frac{\Theta_n[w(C_{j+1})] - \Theta_n[w(C_j)]}{l} \tag{49}$$

$$\left.\frac{\partial^2 w}{\partial t^2}\right|_{s_i^+} \cong \frac{w(C_{j+2}) - 2w(C_{j+1}) + w(C_j)}{l^2} \tag{50}$$

where s_i^+ denotes the arc abscissa at the right-hand side of the corner A_i. By substituting (48), (49) and (50) into expression (46), the twisting moment $T_n(w)$ at a corner point A_i $(i = 1, \ldots, N)$ can be finally estimated by the nodal values of $M_n(w)$, $\Theta_n(w)$ and w in the neighborhood or this corner point.

4. MATRIX FORMULATION

The numerical formulations of the flexural vibration and buckling problems are modelled in this paper by a relatively unsophisticated discretization scheme. The plate boundary is partitioned with a succession of k straight elements K_i $(i = 1, \ldots, k)$, care being taken to locate each corner point A_i $(i = 1, \ldots, N)$ on Γ at a junction point of two consecutive straight boundary elements. On the other hand, the plate domain is divided into m rectangular surface elements M_j $(j = 1, \ldots, m)$. In practice, the discretization of the interior domain is carried out generally with reference to that of the boundary, so that the plate interior domain might be approximated as best as possible by a finite number of rectangular elements.

Only one node C_i $(i = 1, \ldots, k)$ is defined at the middle point of each boundary element K_i. the fundamental boundary variables w, $\Theta_n(w)$, $M_n(w)$ and $V_n(w)$ are supposed to be constant along every straight element K_i $(i = 1, \ldots, k)$, their values being those taken by them at the node C_i. The nodal values of these boundary variables form the following vectors

$$\{\mathbf{w}\} = [w(C_1), \ldots, w(C_k)]^T \tag{51}$$

$$\{\Theta_n\} = [\Theta_n(C_1), \ldots, \Theta_n(C_k)]^T \tag{52}$$

$$\{\mathbf{M}_n\} = [M_n(C_1), \ldots, M_n(C_k)]^T \tag{53}$$

$$\{\mathbf{V}_n\} = [V_n(C_1), \ldots, V_n(C_k)]^T. \tag{54}$$

Similarly, we define only one node G_j $(j = 1, \ldots, m)$ at the center of each domain element M_j. The deflection $w(x, y)$ and the operator $\mathbb{P}(w)$ inside the domain Ω are supposed to be constant over every domain element M_j ($j = 1, \ldots, m$), their values being equal to those taken by them at the node G_j ($j = 1, \ldots, m$) of this element. Two corresponding vectors can be defined as

$$\{\mathbf{w}_s\} = [w(G_1), \ldots, w(G_m)]^T \tag{55}$$

$$\{\mathbf{p}\} = [\mathbb{P}(G_1), \ldots, \mathbb{P}(G_m)]^T \tag{56}$$

where the subscript of $\{\mathbf{w}_s\}$ indicates that the deflection values are related here to the plate domain Ω.

By placing successively the source point $P \in \Gamma$ at the k nodes of the discretized boundary, one can establish the boundary integral equations (23) and (24) in the following matrix form

$$(\tfrac{1}{2}[\mathbf{I}]+[\mathbf{A}_1])\{\mathbf{w}\}+[\mathbf{B}_1]\{\Theta_n\}+[\mathbf{C}_1]\{\mathbf{M}_n\}+[\mathbf{D}_1]\{\mathbf{V}_n\}+\{\mathbf{t}_1\} = \lambda[\mathbf{E}_1]\{\mathbf{p}\} \tag{57}$$

$$[\mathbf{A}_2]\{\mathbf{w}\}+(\tfrac{1}{2}[\mathbf{I}]+[\mathbf{B}_2])\{\Theta_n\}+[\mathbf{C}_2]\{\mathbf{M}_n\}+[\mathbf{D}_2]\{\mathbf{V}_n\}+\{\mathbf{t}_2\} = \lambda[\mathbf{E}_2]\{\mathbf{p}\} \tag{58}$$

where $[\mathbf{I}]$ is the $k \times k$ unit matrix; $[\mathbf{A}_1]$, $[\mathbf{B}_1]$, $[\mathbf{C}_1]$, $[\mathbf{D}_1]$ and $[\mathbf{A}_2]$, $[\mathbf{B}_2]$, $[\mathbf{C}_2]$, $[\mathbf{D}_2]$ are eight $k \times k$ square matrices whose coefficients are calculated from the curvilinear integrals encountered in the boundary integral eqns (23) and (24); $[\mathbf{E}_1]$ and $[\mathbf{E}_2]$ are two $k \times m$ matrices whose coefficients are obtained from the surface integrals of (23) and (24); and $\{\mathbf{t}_1\}$, $\{\mathbf{t}_2\}$ represent the sum of discontinuity jumps encountered in (23) and (24), respectively. By using the technique described above for the treatment of twisting moment $T_n(w)$, one can express the vectors $\{\mathbf{t}_1\}$ and $\{\mathbf{t}_2\}$ as

$$\{\mathbf{t}_\alpha\} = [\mathbf{A}'_\alpha]\{\mathbf{w}\}+[\mathbf{B}'_\alpha]\{\Theta_\alpha\}+[\mathbf{C}'_\alpha]\{\mathbf{M}_n\} \quad (\alpha = 1 \quad \text{and} \quad 2) \tag{59}$$

where the $k \times k$ matrices $[\mathbf{A}'_\alpha]$, $[\mathbf{B}'_\alpha]$ and $[\mathbf{C}'_\alpha]$, with $\alpha = 1$ and 2, can be obtained by employing the approximations (48)–(50) for each corner point A_i ($i = 1, \ldots, N$) of boundary Γ Substituting (59) into (57) and (58), we obtain

$$[\mathbf{A}''_1]\{\mathbf{w}\}+[\mathbf{B}''_1]\{\Theta_n\}+[\mathbf{C}''_1]\{\mathbf{M}_n\}+[\mathbf{D}_1]\{\mathbf{V}_n\} = \lambda[\mathbf{E}_1]\{\mathbf{p}\} \tag{60}$$

$$[\mathbf{A}''_2]\{\mathbf{w}\}+[\mathbf{B}''_2]\{\Theta_n\}+[\mathbf{C}''_2]\{\mathbf{M}_n\}+[\mathbf{D}_2]\{\mathbf{V}_n\} = \lambda[\mathbf{E}_2]\{\mathbf{p}\} \tag{61}$$

where

$$\begin{aligned} &[\mathbf{A}''_1] = \tfrac{1}{2}[\mathbf{I}]+[\mathbf{A}_1]+[\mathbf{A}'_1] && [\mathbf{A}''_2] = [\mathbf{A}_2]+[\mathbf{A}'_2] \\ &[\mathbf{B}''_1] = [\mathbf{B}_1]+[\mathbf{B}'_1] && [\mathbf{B}''_2] = \tfrac{1}{2}[\mathbf{I}]+[\mathbf{B}_2]+[\mathbf{B}'_2] \\ &[\mathbf{C}''_1] = [\mathbf{C}_1]+[\mathbf{C}'_1] && [\mathbf{C}''_2] = [\mathbf{C}_2]+[\mathbf{C}'_2]. \end{aligned} \tag{62}$$

At each node of the boundary, two corresponding values of $\{\mathbf{w}\}$, $\{\Theta_n\}$, $\{\mathbf{M}_n\}$ and $\{\mathbf{V}_n\}$ should vanish because of the two homogeneous boundary conditions chosen among (13)–(15) following on the type of the problem. Hence for a discretization of the boundary into k straight elements, there are in total $2k$ boundary unknowns, every node containing two unknowns defined by two of the fundamental variables $\{\mathbf{w}\}$, $\{\Theta_n\}$, $\{\mathbf{M}_n\}$ and $\{\mathbf{V}_n\}$. By removing, in (60) and (61), the terms associated with the homogeneous boundary conditions of the problem, one can finally obtain the following system of $2k$ linear equations

$$[\mathbf{G}]\{\mathbf{X}\} = \lambda[\mathbf{H}]\{\mathbf{p}\} \tag{63}$$

where the column-vector $\{\mathbf{X}\}$ contains the $2k$ conventional boundary unknowns determined from $\{\mathbf{w}\}$, $\{\Theta_n\}$, $\{\mathbf{M}_n\}$ and $\{\mathbf{V}_n\}$; $[\mathbf{G}]$ is a $2k \times 2k$ matrix derived from the matrices $[\mathbf{A}''_\alpha]$,

$[B''_\alpha]$, $[C''_\alpha]$ and $[D_\alpha]$ ($\alpha = 1, 2$) in (60) and (61); and the $2k \times m$ matrix [H] is obtained by $[E_1]$ and $[E_2]$ in (60) and (61).

In a similar manner, one can write the integral representation (26) of the operator $\mathbb{P}(w)$, by locating successively the source point $P \in \Omega$ at the m nodes G_j ($j = 1, \ldots, m$) of the domain mesh, in the following matrix form

$$\{p\} = [G_p]\{X\} + \lambda[H_p]\{p\} \tag{64}$$

where $[G_p]$ is a $m \times 2k$ matrix whose coefficients are calculated, in a similar way as for obtaining the matrix [G] in (63), from the curvilinear integrals of integral representation (26); and $[H_p]$ is a $m \times m$ matrix whose coefficients are calculated from the surface integrals encountered in (26).

To eliminate the $2k$ conventional boundary unknowns $\{X\}$ in the matrix eqns (63) and (64), one can rewrite (63) as follows

$$\{X\} = \lambda[G]^{-1}[H]\{p\} \tag{65}$$

where $[G]^{-1}$ is the inverse matrix of [G]. By substituting (65) into (64), one obtains the following equation

$$\{p\} = \lambda([G_p][G]^{-1}[H] + [H_p])\{p\} \tag{66}$$

which can be condensed in the following form

$$[F]\{p\} = \frac{1}{\lambda}\{p\} \tag{67}$$

where the square matrix [F] of $m \times m$ is given by

$$[F] = [G_p][G]^{-1}[H] + [H_p]. \tag{68}$$

In view of the matrix equation (67), we observe that the problem under consideration is finally reduced into an eigenvalue problem of the square matrix [F]. The eigenvalues of [F] ($1/\lambda$) correspond to the frequencies of free vibration, or to the critical loads of the buckling problem, and the corresponding eigenvectors of [F] (vector $\{p\}$) represent the mode shapes of deflection (for vibration), or of the curvature (for the buckling) of the problem.

In the case of the buckling problem, the mode shapes of deflection $\{w_s\}$ (the deflection eigenvectors), corresponding to the critical load $\lambda = N_{cr}$, can be obtained by using the integral representation (25). Like that for obtaining the matrix eqn (64), one can express the integral representation (25) in a similar matrix form as follows

$$\{w_s\} = [G_s]\{X\} + \lambda[H_s]\{p\} \tag{69}$$

where $[G_s]$ is a $m \times 2k$ matrix obtained from the curvilinear integrals of (25); and $[H_s]$ is a $m \times m$ matrix derived from the surface integrals of (25). By substituting (65) into (69), we obtain finally

$$\{w_s\} = \lambda([G_s][G]^{-1}[H] + [H_s])\{p\}. \tag{70}$$

As λ and $\{p\}$ have been calculated by the solution of the above eigenvalue problem (67), it is then possible to calculate, by using (70), the deflection values $\{w_s\}$ at the m nodal points G_j ($j = 1, \ldots, m$) inside the plate domain. These deflection values $\{w_s\}$ define effectively the mode shapes of the orthotropic plate in equilibrium under the corresponding critical load N_{cr}.

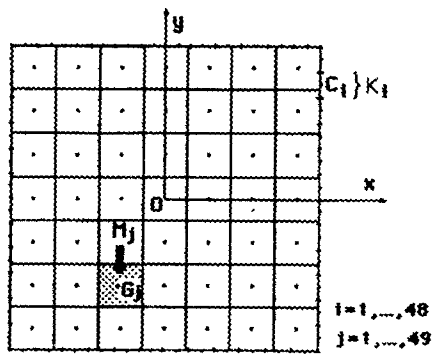

Fig. 2. Discretization of a square plate.

5. NUMERICAL RESULTS

We have solved, using the DBEM technique proposed above, some example problems of the free flexural vibration and buckling of square or rectangular orthotropic plates. Different boundary conditions imposed on the four edges of the plates are considered, including the simply-supported edges, the clamped edges, and the mixed edge conditions, etc. The orthotropic plates considered in the following have, in general, the elastic properties corresponding to the graphite/epoxy material, except in some examples where the plates are made up of special materials. The elastic constants of a graphite/epoxy material, with the fibers directed in the x-axis direction, are given as follows (Tsai and Hahn, 1980):

$$E_1 = 181\,\text{GPa} \qquad E_2 = 10.3\,\text{GPa}$$
$$G = 7.17\,\text{GPa} \qquad \nu_1 = 0.28. \tag{71}$$

The flexural rigidities of such a graphite/epoxy plate with thickness $h = 0.01$ m can be calculated by using (1).

The discretization of the plate boundary and its interior domain is performed with reference to the plan forms of the plates. For example, in the case of a square plate, the boundary is partitioned in such a way that each of the four edges be divided into 12 rectilinear segments of equal length, hence 48 boundary elements in total for all the contour of the plate (Fig. 2). And the interior domain of the square plate is discretized into a mesh of $7 \times 7 = 49$ panels (Fig. 2). The discretization of rectangular plates could be carried out just as easily in the similar way.

5.1. *Flexural vibration problems*

For the purpose of simplicity, numerical results for the free vibration frequencies ω will be presented in dimensionless form as follows

$$K_v = \frac{\omega a^2}{\sqrt{D_3/\rho h}} \tag{72}$$

where K_v is defined as a frequency parameter, and a represents the edge length of the plate along the x-axis direction.

Example 1: *Simply-supported or clamped rectangular graphite/epoxy plates.* The first problem considered is that of the free vibration of rectangular graphite/epoxy plates with their four edges simply-supported or clamped. The numerical computation is emphasized in the calculation of fundamental (first vibration mode) frequencies of rectangular plates with different edge ratios $c = a/b$, where a and b are the edge lengths along the x- and y-axis, respectively. Such a problem was studied by Lekhnitskii (1968) using the Fourier method in the case of simply-supported edges, and the simplified Rayleigh–Ritz method in

Table 1. Fundamental frequency parameter K_v of rectangular graphite/epoxy plates

Edge ratio c		0.5	1.0	1.5	2.0
	DBEM	33.198	36.044	42.426	53.046
Simply supported	Fourier	32.861	35.787	41.947	52.366
	Error (%)	1.03	0.72	1.08	1.30
	DBEM	74.170	77.418	87.286	107.524
Clamped	Rayleigh–Ritz	73.532	76.854	86.575	106.337
	Error (%)	0.87	0.73	0.82	1.12

that of clamped edges. The series-type solution (simply-supported plates) and the first-order approximation (clamped plates) given by Lekhnitskii can be used to calculate the fundamental frequencies of the graphite/epoxy plate.

Applying the present DBEM, we have calculated the fundamental frequencies of the rectangular plates with various edge ratios ($c = 0.5$, 1.0, 1.5 and 2.0). The numerical results obtained for the frequency parameter K_v defined in (72), are presented in Table 1. As shown in this Table, the values of K_v obtained by the present DBEM are in excellent agreement with the results available from the Fourier method, or the Rayleigh–Ritz method (Lekhnitskii, 1968), since the relative error between our DBEM and the approximate method used by Lekhnitskii is only about 1.0% for both the simply supported and the clamped plates.

Example 2: *Square orthotropic plate clamped along two opposed edges and free along other two edges.* This example is selected from Dickinson (1969), in order to test the performance of the proposed DBEM for the solution of free vibration problems in the case of mixed boundary conditions. The flexural rigidities of the plate are those used by Dickinson (1969):

$$D_{11}/D_3 = 1.543 \quad D_{22}/D_3 = 4.81 \quad \nu_1 = 0.039. \tag{73}$$

Along the four edges of the square plate, the boundary conditions are applied as follows:

—clamped along the two opposed edges parallel to the y-axis;
—free along the other two edges parallel to the x-axis (see Fig. 2).

The frequencies of the five first vibration modes are calculated using the current DBEM. The numerical results obtained for the frequency parameter K_v, defined in (72), are given in Table 2.

A comparison of our numerical results with those given by Dickinson (1969) show a good precision of the present DBEM, and the relative error for the values of frequency parameter K_v between the two methods varies from 1.0 to 4.0% for the five first vibration modes.

Example 3: *Square cantilever graphite/epoxy plate.* For the vibration problem of a cantilever orthotropic plate, we do not know a comparative solution available in the literature, so we give only the numerical results obtained by the proposed DBEM.

The boundary edge $x = a/2$ (see Fig. 2) of the square plate is clamped, and the other three edges are all free. We have calculated, using our DBEM, the frequencies of such a

Table 2. Frequency parameter K_v of an orthotropic plate with mixed boundary conditions

Mode ($m \times n$)	DBEM	Dickinson	Error (%)
1 (1×1)	27.957	27.730	0.88
2 (1×2)	32.139	31.634	1.57
3 (1×3)	66.783	64.558	3.45
4 (2×1)	78.345	76.404	2.54
5 (2×2)	85.251	81.774	4.25

Table 3. Frequency parameter K_v of square cantilever graphite/epoxy plate

Mode $(m \times n)$	$1 (1 \times 1)$	$2 (1 \times 2)$	$3 (1 \times 3)$	$4 (1 \times 4)$	$5 (2 \times 1)$
DBEM	11.512	14.846	28.924	60.301	73.319

cantilever plate, and also the corresponding mode shapes. The numerical results for the frequency parameter K_v, defined in (72), are presented in Table 3 for the first five vibration modes, and the mode shapes (deflection) from the first to the fourth vibration mode are given in Fig. 3. As shown in this figure, the first and the third mode shapes correspond to a symmetric deformation of the cantilever plate with respect to the x-axis; but the second and the fourth to an anti-symmetric one.

5.2. *Buckling problems*

The numerical results obtained for the buckling problems of orthotropic plates will be presented in the following in terms of the dimensionless parameter K_b which is defined by

$$K_b = \frac{N_{cr} a^2}{D_3} \tag{74}$$

where N_{cr} is the critical load, and a represents the edge length of the plate along the x-axis direction.

The computational examples concern some square or rectangular orthotropic plates with different boundary conditions (simply-supported, clamped, mixed). The buckling loadings (in-plane forces) N_x, N_{xy} and N_y are represented by the factors a_x, a_{xy} and a_y, as defined in (6). The signs of a_x, a_{xy} and a_y depend on the type of in-plane forces N_x, N_{xy} and N_y. For example, if N_x is a tensile force, $a_x > 0$; and if N_x is of compression, then $a_x < 0$.

Example 4: *Simply-supported square plate under different loadings*. In this example, we have applied the proposed DBEM to the solution of buckling problem of a square birch plywood plate simply-supported along the four edges. The flexural rigidities of the plate are (Lekhnitskii, 1968):

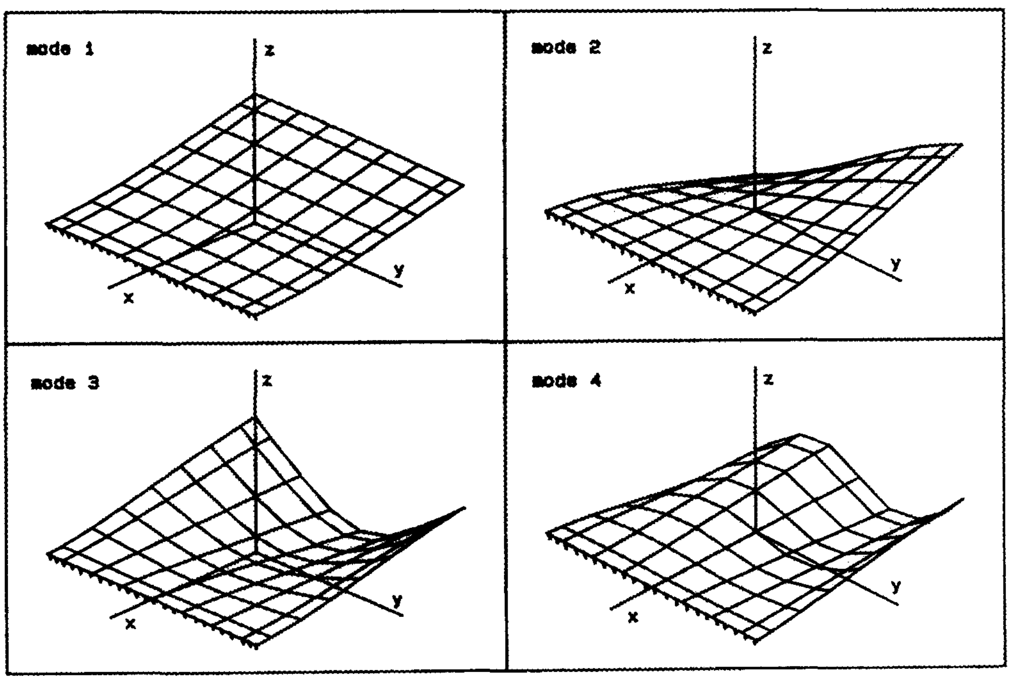

Fig. 3. Vibration mode shapes (deflection) of a square cantilever graphite/epoxy plate.

Table 4. Critical load parameter K_b of a simply-supported birch plywood plate under different loadings

Loadings a_x	a_{xy}	a_y	Critical load parameter K_b DBEM	Lekhnitskii	Error (%)
−1	0	0	72.828	71.457	1.92
0	−1	0	158.039	156.359	1.07
0	0	−1	49.117	47.587	3.21
−1	0	−1	36.449	35.728	2.02
−1	−1	−1	32.687	—	—
−1	0	1	65.293	63.450	2.90
−1	0	−2	21.612	21.149	2.19

$$D_{11} = 0.11876 \times 10^5 \text{Nm} \qquad D_{22} = 0.98967 \times 10^3 \text{Nm}$$
$$D_{12} = 0.45525 \times 10^3 \text{Nm} \qquad D_{66} = 0.10000 \times 10^4 \text{Nm}. \tag{75}$$

The plate is subjected to different in-plane forces (uniaxial or biaxial compression, uniform shear force, and mixed compression-tension forces, etc.) along the four edges. For the buckling of such a plate under compression or/and tension forces, an exact solution is available for the calculation of critical loads N_{cr} (Lekhnitskii, 1968). In the case where the plate is subjected to a uniform shear force along its four edges, some results for the critical loads were obtained by using the Rayleigh–Ritz method (Lekhnitskii, 1968). In Table 4, we give the results obtained for the critical load parameter K_b, defined in (74), of the square birch plywood plate under different loadings.

As shown in Table 4, the results of the parameter K_b, obtained by our DBEM, differ only about 3% from the values calculated by the exact solution (compression, tension) or the Rayleigh–Ritz method (shear force).

Example 5: *Square graphite/epoxy plate with different boundary conditions.* Two problems are treated in this example, concerning a square graphite/epoxy plate subjected to a uniaxial compression:

—plate clamped along the four edges;

—plate simply-supported along the two opposed edges parallel to y-axis, and clamped along the two other edges parallel to x-axis.

The uniaxial compression is uniformly distributed along the two opposed plate sides in the x- and y-axis direction, respectively. The elastic constants of the graphite/epoxy plate have been given in (71).

We present, in Table 5, the numerical results of the critical load parameter K_b, defined in (74), for the graphite/epoxy plate. In the case of the plate with two simply-supported and two clamped edges, we have compared our results with those given by the Rayleigh–Ritz method (Lekhnitskii, 1968).

Example 6: *Simply-supported rectangular graphite/epoxy plates under uniaxial compression.* This example consists of studying the variations of the critical load N_{cr} and of the deflection mode shapes following the edge length ratios of a rectangular graphite/epoxy plate simply-supported along its four edges. The fibers of the graphite/epoxy material are directed in the y-axis direction. The corresponding flexural rigidities of the plate, with thickness $h = 0.01$ m, are given by:

Table 5. Critical load parameter K_b for a graphite/epoxy plate with different boundary conditions

	Loadings	DBEM	Lekhnitskii	Error (%)
Four clamped edges	$a_x \neq 0$	481.213	—	—
	$a_y \neq 0$	168.161	—	—
Two clamped and two simply-supported edges	$a_x \neq 0$	163.239	162.017	0.75
	$a_y \neq 0$	143.894	141.326	1.82

Table 6. Values of K_b for simply-supported rectangular graphite/epoxy plates under uniaxial compression

Ratio c	Mode ($m \times n$)	DBEM	Lekhnitskii	Error (%)
0.5	1 × 1	17.806	17.365	2.54
1.0	2 × 1	71.358	69.461	2.73
1.5	3 × 1	161.714	156.288	3.47
2.0	4 × 1	291.226	277.845	4.82

$$D_{11} = 0.86218 \times 10^3\,\text{Nm} \qquad D_{22} = 0.15151 \times 10^5\,\text{Nm}$$

$$D_{12} = 0.24141 \times 10^3\,\text{Nm} \qquad D_{66} = 0.59750 \times 10^3\,\text{Nm}. \tag{76}$$

Suppose that the plate is subjected to a uniaxial compression force in the x-axis direction ($a_x \neq 0$). The edge length ratio of the rectangular plate is denoted by $c = a/b$, where a and b represent respectively the two edge lengths of the rectangular plate.

In Table 6, we compare the results of parameter K_b, defined in (74) for the critical load N_{cr}, obtained by the proposed method with those calculated from the exact solution (Lekhnitskii, 1968).

Figure 4 shows the buckling deflection mode shapes of the simply-supported rectangular graphite/epoxy plate with edge ratios $c = 0.5$, 1.0, 1.5 and 2.0. One observes in this figure that for $c = 0.5$, the plate undergoes a unilateral buckling deformation; but for the edge ratio $c = 1.0$, 1.5 and 2.0, the graphite/epoxy plate is in a bilateral buckling deformation under the corresponding critical load N_{cr}.

6. CONCLUSIONS

The computational examples treated in this paper clearly illustrate the efficiency and the versatility of the proposed direct boundary element method for the solution of flexural vibration and buckling problems of orthotropic plates. One of the essential characteristics of this numerical technique is the fact that the fundamental solution for the bending of orthotropic plates is employed in the integral formulations of the vibration and the buckling

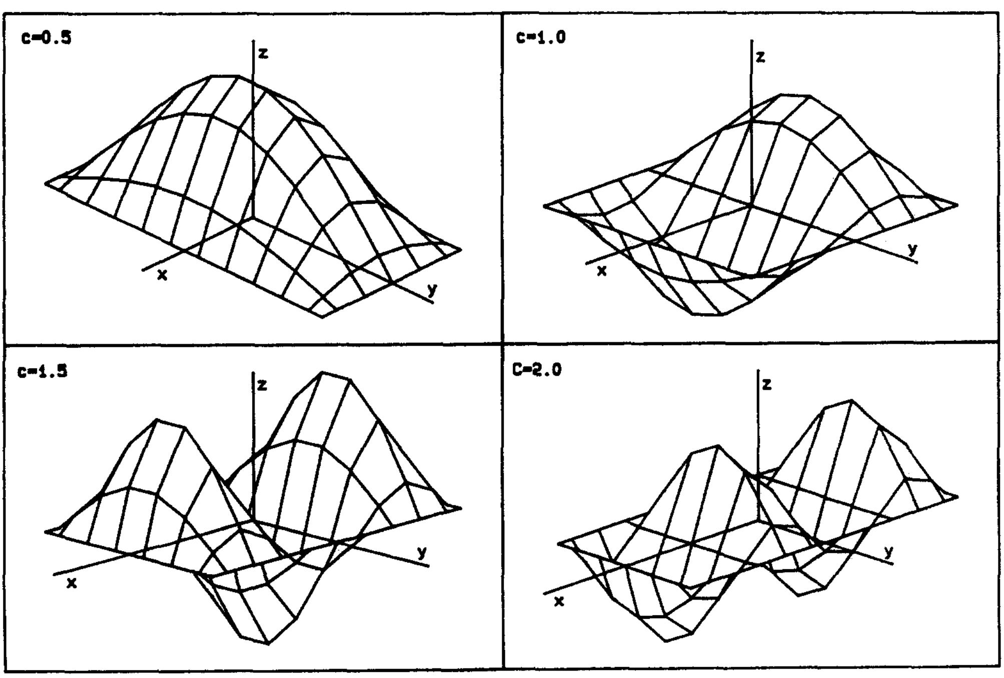

Fig. 4. Buckling mode shapes (deflection) of simply-supported rectangular graphite/epoxy plates.

analysis. As the boundary integral equations involve the conventional physical quantities of Kirchhoff's theory, all the usual boundary conditions in practice could be easily taken into account, even the mixed boundary conditions. Moreover, by reason of the treatment of twisting moment at the corners of the plate boundary, the present method can be efficiently applied to analyse the vibration or the buckling problem of any orthotropic plates, whatever their geometric plan forms.

It has been demonstrated that the numerical results obtained using the current direct BEM are in good agreement with the results available from the analytic solutions or the Rayleigh–Ritz method for both the calculation of the vibration frequencies and the determination of the buckling critical loads of orthotropic plates.

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APPENDIX

As indicated in the case of anisotropic plate bending (Shi and Bézine, 1988), the derivatives of the fundamental solution (33) can be expressed in terms of the corresponding derivatives of the functions $R_i(Q;P)$ and $S_i(Q;P)$ given in (34) and (35), where Q and P are respectively the distribution point and the source point of coordinates (x, y) and (x_0, y_0). In consequence, only the fifth-order derivatives of the functions $R_i(Q;P)$ and $S_i(Q;P)$ will be given in the following.

—Fifth-order derivatives of $R_i(Q;P)$ with $i = 1$ and 2:

$$\frac{\partial^5 R_i}{\partial x^5} = \frac{-8}{(x_i^2+y_i^2)^3}[-x_i^3+3x_iy_i^2]$$

$$\frac{\partial^5 R_i}{\partial x^4 \partial y} = \frac{-8}{(x_i^2+y_i^2)^3}[-d_ix_i^3-3e_ix_i^2y_i+3d_ix_iy_i^2+e_iy_i^3]$$

$$\frac{\partial^5 R_i}{\partial x^3 \partial y^2} = \frac{-8}{(x_i^2+y_i^2)^3}[(e_i^2-d_i^2)x_i^3-6d_ie_ix_i^2y_i+3(d_i^2-e_i^2)x_iy_i^2+2d_ie_iy_i^3]$$

$$\frac{\partial^5 R_i}{\partial x^2 \partial y^3} = \frac{-8}{(x_i^2+y_i^2)^3}[d_i(3e_i^2-d_i^2)x_i^3+3e_i(e_i^2-3d_i^2)x_i^2y_i+3d_i(d_i^2-3e_i^2)x_iy_i^2+e_i(3d_i^2-e_i^2)y_i^3]$$

$$\frac{\partial^5 R_i}{\partial x \partial y^4} = \frac{-8}{(x_i^2+y_i^2)^3}[(-d_i^4+6d_i^2e_i^2-e_i^4)x_i^3+12d_ie_i(e_i^2-d_i^2)x_i^2y_i+3(d_i^4-6d_i^2e_i^2+e_i^4)x_iy_i^2+4d_ie_i(d_i^2-e_i^2)y_i^3]$$

$$\frac{\partial^5 R_i}{\partial y^5} = \frac{-8}{(x_i^2+y_i^2)^3}[d_i(-d_i^4+10d_i^2e_i^2-5e_i^4)x_i^3+3e_i(-5d_i^4+10d_i^2e_i^2-e_i^4)x_i^2y_i$$
$$+3d_i(d_i^4-10d_i^2e_i^2+5e_i^4)x_iy_i^2+e_i(5d_i^4-10d_i^2e_i^2+e_i^4)y_i^3].$$

—Fifth-order derivatives of $S_i(Q;P)$ with $i = 1$ and 2:

$$\frac{\partial^5 S_i}{\partial x^5} = \frac{-4}{(x_i^2+y_i^2)^3}[3x_i^2y_i-y_i^3]$$

$$\frac{\partial^5 S_i}{\partial x^4 \partial y} = \frac{-4}{(x_i^2+y_i^2)^3}[-e_ix_i^3+3d_ix_i^2y_i+3e_ix_iy_i^2-d_iy_i^3]$$

$$\frac{\partial^5 S_i}{\partial x^3 \partial y^2} = \frac{-4}{(x_i^2+y_i^2)^3}[-2d_ie_ix_i^3+3(d_i^2-e_i^2)x_i^2y_i+6d_ie_ix_iy_i^2+(e_i^2-d_i^2)y_i^3]$$

$$\frac{\partial^5 S_i}{\partial x^2 \partial y^3} = \frac{-4}{(x_i^2+y_i^2)^3}[e_i(-3d_i^2+e_i^2)x_i^3+3d_i(d_i^2-3e_i^2)x_i^2y_i+3e_i(3d_i^2-e_i^2)x_iy_i^2+d_i(-d_i^2+3e_i^2)y_i^3]$$

$$\frac{\partial^5 S_i}{\partial x \partial y^4} = \frac{-4}{(x_i^2+y_i^2)^3}[4d_ie_i(-d_i^2+e_i^2)x_i^3+3(d_i^4-6d_i^2e_i^2+e_i^4)x_i^2y_i+12d_ie_i(d_i^2-e_i^2)x_iy_i^2+(-d_i^4+6d_i^2e_i^2-e_i^4)y_i^3]$$

$$\frac{\partial^5 S_i}{\partial y^5} = \frac{-4}{(x_i^2+y_i^2)^3}[e_i(-5d_i^4+10d_i^2e_i^2-e_i^4)x_i^3+3d_i(d_i^4-10d_i^2e_i^2+5e_i^4)x_i^2y_i$$
$$+3e_i(5d_i^4-10d_i^2e_i^2+e_i^4)x_iy_i^2+d_i(-d_i^4+10d_i^2e_i^2-5e_i^4)y_i^3]$$

where x_i and y_i are defined in (36) by the coordinates of the distribution point $Q(x, y)$ and of the source point $P(x_0, y_0)$.

An analogical derivation process can be carried out to obtain the corresponding derivatives of the fundamental solution (40) in the case of $D_3^2 = D_{11}D_{22}$. It should also be noted here that the source point $P(x_0, y_0)$ in the integral representation (26) is always situated inside the plate domain Ω, so there are no third-order singularities of the boundary integrals in the integral formulation for the buckling analysis, even though the fifth-order derivatives given above present in such forms.